The Probability Approach to Default Probabilities

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Abstract

The probability approach to uncertainty and modeling is applied to default probability estimation. This issue has attracted attention as banks contemplate the requirements of Basel II's IRB rules. Nicholas M. Kiefer proposes the fomal introduction of expert information into quantitative analysis. An application treating the incorporation of expert information on the default probability is considered in detail.

Keywords: Bayesian inference, Bayesian estimation, expert information, Basel II, risk management

JEL Classifications: C11, C13, C44, G18, G32

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1 Introduction

Estimation of default probabilities (PD), loss given default (LGD, a fraction) and exposure at default (EAD) for portfolio segments containing reasonably homogeneous assets is essential to prudent risk management as well as for compliance with Basel II rules for banks using the IRB approach to determine capital requirementsBasel Committee on Banking Supervision (2004). Estimation of small probabilities is tricky, and I will focus on estimating PD. This problem has attracted considerable recent attention; see Basel Committee on Banking Supervision (2005), Balthazar (2004), BBA, LIBA, and ISDA (2005),and Pluto and Tasche (2005). This is an application in which data information is scarce while expert information is available. The focus of this paper is on estimation of the default probability for a risk bucket on the basis of historical information and expert knowledge. Section 2 argues for the probability approach to uncertainty measurement. The probability approach to default modeling is uncontroversial, although perhaps the extent of the constraints imposed by the simple independent Bernoulli model are underappreciated. This model is briefly described in Section 3. In section 4 we argue that exactly the same considerations that lead to the probability approach for defaults should lead to the probability approach to default probabilities. As an example, we consider describing expert information in the form of a Beta distribution on the Bernoulli parameter. The probability approach allows coherent combination of expert and data information through Bayes Rule, taken up in Section 5. Section 6 considers estimators of PD based on the probability approach and compares them with alternatives, including the maximum likelihood estimator (which is also the unbiased estimator) and a recent suggestion based on the upper endpoint of a confidence interval of prespecified coverage.

2 The Probabilistic Characterization of Uncertainty

There are a number of arguments that uncertainty is best described in terms of probabilities. These can be based on prediction scoring, avoidance of sure losses in betting, pareto optimality, etc. Important references are De Finetti (1974), Lindley (1982), and Savage (1954). These arguments lead to a requirement of *coherence*. This weak requirement is just that systems of numbers describing uncertainty will not be such that another system can beat them in prediction, or that, if used for betting, they will not admit sure losses. This simple requirement is enough to insure that the predictions must combine like probabilities. Thus, let E, F, denote events (e.g. "asset 2 and only asset 2, defaults"). Let x, y, z be numbers used to quantify the uncertainty about events. The three properties implied by coherence are:

P1: Convexity: $0 \leq x \leq 1$.

P2: Additivity: Let x refer to the event E and y the event $\sim E$. Then $x + y = 1$.

P3: Multiplication: Let x correspond to E, y to F given E, and z to E and F. Then $z = xy$.

These three properties are often taken as defining a system of probabilities.

Of course, the probability approach to describing and modeling default uncertainty is central to risk management and to the requirements of Basel II. In the case of default modeling, where measuring and controlling risk is the aim, it is widely accepted that the probability approach is the correct approach to default uncertainty. There is no serious argument that the probability approach is wrong or inappropriate for modeling uncertain future defaults as well as other unknowns. The fact that probabilities combine in accordance with convexity, additivity and multiplication is central for moving from probabilities of default on an asset, to default rates in a segment, to rates in a portfolio, and to a default probability for the bank. It is less well accepted that uncertainty about the unknown default probability can be usefully modeled in exactly the same way for exactly the same reasons.

3 A Statistical Model for Uncertain Defaults

The requirement of coherence prescribes relations among the probabilities of related events, but does not specify what these probabilities are. The usual approach in statistical modeling is to choose a statistical model that generates all the relevant probabilities as a function of a small number of parameters. The simplest and most common probability model for defaults of assets in a homogeneous segment of a portfolio is the Bernoulli, in which the defaults are independent across assets and over time, and defaults occur with common probability θ. Note that specification of this model requires expert judgement, that is, information. We will denote the expert information by e. The role of expert judgement is not usually explicitly indicated at this stage, so it is worthwhile to point to its contribution. First, consider the statistical model. The independent Bernouilli model is not the only possibility. Certainly independence is a strong assumption and would have to be considered carefully. Second, are the observations really identically distributed? Perhaps the default probabilities differ across assets. Can this be modeled, perhaps on the basis of asset characteristics? The usual requirements demand an annual default probability, estimated over a sample long enough to cover a full cycle of economic conditions. Thus the probability should be marginal with respect to external conditions. For specificity we will continue with the Bernoulli specification. Let d_i indicate whether the ith observation was a default ($d_i = 1$) or not ($d_i = 0$). The Bernoulli model for the distribution of d_i is $p(d_i|\theta, e) = \theta^{d_i}(1-\theta)^{1-d_i}$. Let $D = \{d_i, i = 1, ..., n\}$ denote the whole data set and $r = r(D) = \sum_i d_i$ the count of defaults. Then the joint distribution of the data is

$$\begin{aligned} p(D|\theta, e) &= \prod \theta^{d_i}(1-\theta)^{1-d_i} \\ &= \theta^r(1-\theta)^{n-r} \end{aligned} \tag{3.1}$$

As a function of θ for given data D this is the likelihood function $L(\theta|D, e)$. Since this distribution depends on the data D only through r (n is regarded as fixed), the sufficiency principle implies that we can concentrate attention on the distribution of r

$$p(r|\theta, e) = \binom{n}{r}\theta^r(1-\theta)^{n-r} \tag{3.2}$$

a Binomial (n, θ) distribution. This is a tremendous simplification, since, for fixed n, there are only a small number of likely defaults r, so analysis can be done for all likely datasets.

4 A Statistical Model for Uncertain Default Probabilities

Equation 3.2 is a model for describing default probabilities (probabilities for different default configurations in a portfolio segment), but it is an incomplete model in that the parameter θ remains unspecified. The default probability θ is an unknown, but that doesn't mean that nothing is known about its value. In fact, defaults are widely studied and risk managers, modelers, validators, and supervisors have detailed knowledge on values of θ for particular portfolio segments. The point is that θ is unknown in the same sense that the future default status of a particular asset is unknown. We have seen how uncertain defaults can be modeled. The same methods can be used to model the uncertainty about θ. Define events E_i relevant to describing the uncertainty about θ, for example $E_1 =$ "$\theta < .0001$"; E_2 : "$\theta < .0005$," etc. Uncertainty about values of θ are coherently described by probabilities on these events. We assemble these probability assessments into a distribution describing the uncertainty about $\theta, p(\theta|e)$. Our approach is a classical Bayesian approach as described in Raiffa and Schlaifer (1961) .

Now, $p(\theta|e)$ can be a quite general specification, reflecting in general the assessments of uncertainty in an infinity of possible events. This is in contrast with the case of default configurations, in which there are only a finite (though usually large) number of possible default configurations. However, this should not present an insurmountable problem We are quite willing to model the large number of probabilities associated with the possible different default configurations with a simple statistical model; in fact, a 1-parameter model. This involves additional assumptions, including independence, but it simplifies the analysis and allows progress along empirical lines. The same can be done with the prior specification. We can fit a few probability assessments by an expert to a suitable functional form and use that distribution to model prior uncertainty. Of course, as with the likelihood approach, there is some approximation involved, and care is necessary.

A convenient functional form, which we shall carry through this exposition, is given by the beta distribution

$$p(\theta|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \tag{4.1}$$

which has mean $\alpha/(\alpha+\beta)$. variance $\alpha\beta/(\alpha+\beta)^2(1+\alpha+\beta)$ and mode $(\alpha-1)/(\alpha+\beta-2)$. The special case of $\alpha=\beta=1$ is the uniform distribution on the unit interval. Beta-binomial analysis is described in Raiffa and Schlaifer (1961) and DeGroot (1970).

As an illustration, consider a segment of loans which might be in the middle of a bank's portfolio in terms of risk. These loans might be roughly equivalent to S&P Baa or Moody's BBB. Of course, the bulk of these loans are to unrated companies and the bank has done their own rating to assign the loans to risk "buckets.". We have consulted an experienced industry expert on these assets. The expert puts the default probability for assets in this portfolio at 0.01 (an expressed median value). When asked to condition on the probability being less than 0.01, and then considering the conditional median the expert returned the 25% quantile 0.0075. The corresponding question returned the 75% point at 0.0125. Thus this expert reports a rather tight distribution centered on 0.01 and nearly symmetric, probably reflecting extensive experience with portfolios active in this risk segment. We can fit this to a beta distribution, resulting in $\alpha = 6.8$ and $\beta = 647$. This will be a sufficient representation of expert opinion for the point being made in this paper. In practice, the elicitation process is extensive. More information would be extracted from the expert, perhaps wider families of distributions than the beta would be considered, and there would be some iteration back and forth between the statistician and the expert Kiefer (2006)Garthwaite, Kadane, and O'Hagan (2005).

5 Inference

Given the distribution $p(\theta|e)$, we can multiply the probabilities in accord with the multiplication rule to obtain the joint distribution of r, the number of defaults, and θ :

$$p(r, \theta|e) = p(r|\theta, e)p(\theta|e)$$

from which we obtain the marginal (predictive) distribution of r,

$$p(r|e) = \int p(r, \theta|e)d\theta \tag{5.1}$$

If the value of the parameter θ is of main interest (rather than the number of defaults) we can divide to obtain the conditional (posterior) distribution of θ :

$$p(\theta|r, e) = p(r|\theta, e)p(\theta|e)/p(r|e) \tag{5.2}$$

which is Bayes rule DeGroot (1970).
Using specifications 3.2 in which expert opinion appears in the likelihood specification and 4.1 in which expert opinion is reflected in the values of α and β we find for the predictive distribution 5.1

$$p(r|e) = \frac{\Gamma(r+\alpha)\Gamma(n-r+\beta)\Gamma(\alpha+\beta)\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)} \tag{5.3}$$

and for the posterior 5.2

$$p(\theta|r,e) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+r)\Gamma(\beta+n-r)}\theta^{\alpha+r-1}(1-\theta)^{\beta+n-r-1} \tag{5.4}$$

With our example prior distribution for expert opinion the predictive distributions 5.3 of the number of defaults in a portfolio segment of size 100 is plotted in Figure 1.

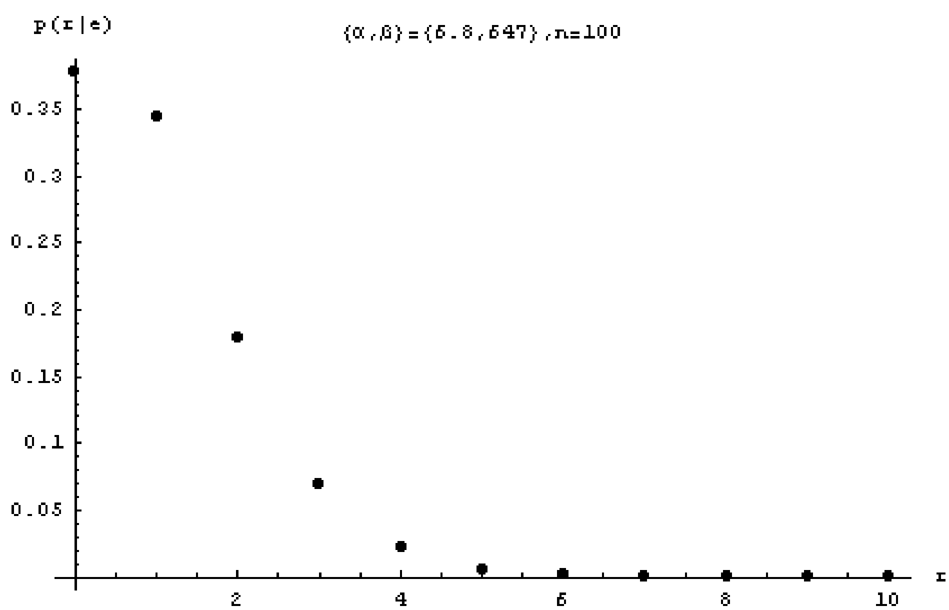

Figure 1: Predictive Distribution $p(r|e)$ for n=100

A candidate estimator for PD, suitable for plugging into the formulas given by the Basel committee's capital model, is

$$\overline{\theta} = E(\theta|r,e) = (\alpha+r)/(\alpha+\beta+n) \tag{5.5}$$

We will not spend much effort justifying the use of the mean; except to point out that the mean is optimal with respect to squared error loss. For any summary statistic, it is appropriate to report an indicator of its reliability, for example the standard deviation

$$\sigma_\theta = \sqrt{E(\theta - E(\theta|r,e))^2} \tag{5.6}$$

The posterior distributions for a sample size of 100 with 0,1, and 5 defaults is plotted in Figure 2. These distributions represent in full the post data uncertainty about θ given the expert information and likely (and some unlikely) samples.

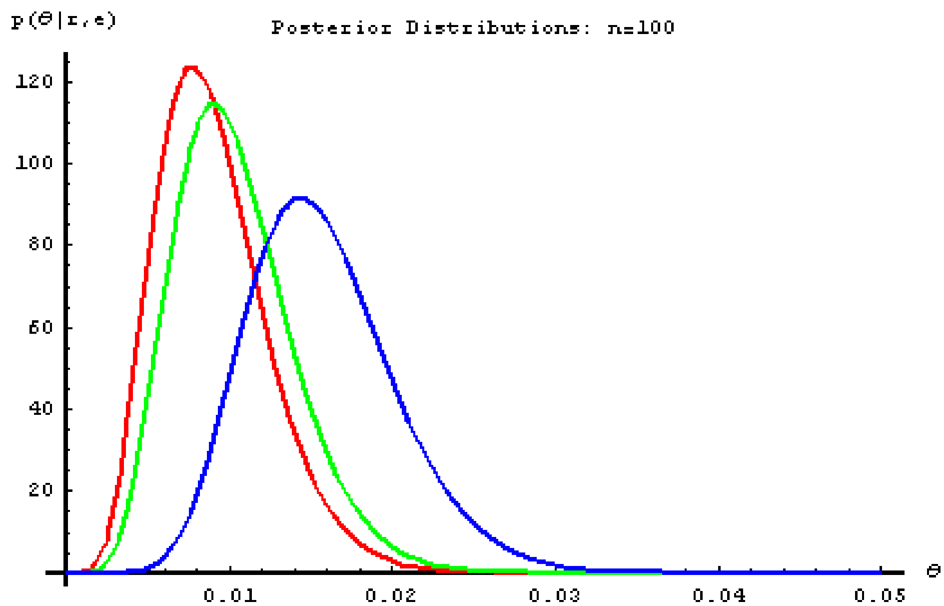

Figure 2: Posterior Distributions for r=0 (red), r=1 (green) and r=5 (blue).

6 Estimators for likely samples

We will advocate estimators based on the probability approach. In the examples tabulated below we will focus on the posterior mean 5.5 as an estimator and the posterior standard deviation 5.6 as a summary of its precision as an estimator. An alternative estimator in wide use and an estimator which may be contemplated by Basel II is the maximum likelihood estimator $\widehat{\theta} = r/n$ with estimated standard error $\sigma_{\widehat{\theta}} = \sqrt{(r/n)(1-r/n)/n}$. This is also the unbiased estimator. For large samples and θ well away from the extremes of 0 and 1, the MLE is a very good estimator in many applications, since the data evidence can be expected to dominate prior evidence provided by the expert. That is, the data evidence is accumulated as the data set increases in size, while the amount of expert evidence is fixed. When that occurs, the likelihood approach and the probability approach coincide as the sample becomes large. Thus, the likelihood estimator can be given a slightly strained probability interpretation. However, for θ near zero (the case in applications we consider) this domination does not occur for any practical sample size Kiefer (2006).

An estimator recently proposed is the confidence estimator Pluto and Tasche (2005). We make our point by considering a simple case of this estimator. The authors give refinements including extensions to simultaneous estimation of default probabilities for different buckets. Our comments apply equally to these cases. The principle attractive feature of the confidence estimator is that it gives a nonzero estimator in the case of zero observed defaults, a case which is extremely likely to occur in low default (probability) portfolio segments. Define F(u|r,n) as the probability of obtaining r or fewer successes in a sequence of n independent Bernoulli trials each with success probability u. Thus

$$F(u|r,n) = \sum_{i=0}^{r} \binom{n}{i} u^i (1-u)^{n-i}$$

Then define the estimator for a prespecified number $\delta \in [0,1]$ chosen by the modeler,

$$\theta_\delta = F^{-1}(\delta|r,n). \tag{6.1}$$

For the case r=0, the estimator takes a simple form, $\theta_\delta = 1 - \delta^{(1/n)}$. A few comments on this procedure are in order. Although we criticize this estimator on both fundamentals and performance, it does represent a major advance in thinking about default probabilities. Specifically, it suggests that the unbiased estimator can be abandoned, and that a practical estimator might adjust the unbiased estimator toward more likely (in this case nonzero) values. One wonders why it is considered easier to think about realistic values for δ than it is to think about θ directly. Pluto and Tasche interpret the choice of δ as a specification of "conservatism." In any case, there does not seem to be a direct probability interpretation of the estimator θ_δ. Rather, θ_δ is interpreted as that value of θ at which the probability of seeing r or fewer defaults is equal to δ. It is difficult to see a justification for this approach, other than the appeal of bounding the estimate away from zero (provided δ, a choice parameter for the modeler, is bounded away from zero). Of course, the posterior mean has a direct probability interpretation. The maximum likelihood estimator has an approximate probability interpretation as well as an interpretation as that value of θ which maximizes the probability of seeing the sample actually observed.

The performance of the alternative estimators is given in Table 1 for n=100 and all plausible values of the number of realized defaults. Note that there is no appealing measure of the sampling variance offered for θ_δ. Results are given based on the posterior distribution using our expert's information; hypothetical information from an alternative, less confident expert with $\alpha = 1.5$, and $\beta = 150$,(this expert has approximately twice the prior standard deviation as our actual expert); from the likelihood alone, and the confidence estimator $\theta_{0.1}$.

Table 1: Estimates and associated standard errors

n	r	$\overline{\theta}$	σ_θ	$\overline{\theta}_{lce}$	$\sigma_{\theta_{lce}}$	$\widehat{\theta}$	$\sigma_{\widehat{\theta}}$	$\theta_{0.1}$
100	0	0.0090	0.0034	0.0060	0.0048	0	0	0.0228
100	1	0.0103	0.0037	0.0099	0.0062	0.01	0.0099	0.0383
100	2	0.0117	0.0039	0.0139	0.0073	0.02	0.0140	0.0523
100	3	0.0130	0.0041	0.0179	0.0083	0.03	0.0171	0.0656
100	4	0.0143	0.0043	0.0219	0.0092	0.04	0.0196	0.0783
100	5	0.0157	0.0045	0.0258	0.0100	0.05	0.0218	0.0908

Notes: The posterior mean and standard deviation, the same based on a less confident expert, the MLE and its s.e., the confidence estimator.

As expected, the posterior mean for our expert is tightly clustered around the prior expectation (about 0.0104) for the sample size 100, even with the unlikely value of 5 defaults. Our hypothetical less-confident expert also supplies estimators which do not seem unreasonable, though of course they are much more sensitive to sample variation. The MLE reflects the well-appreciated problem that the estimator is zero when zero defaults are observed; further the estimated standard error (not an appropriate estimator) is also zero, showing perhaps more confidence than justified. The confidence estimator depends on the user-specified δ; the value chosen 0.10, is suggested in Pluto and Tasche (2005); the values 0.5, 0.25, 0.05, 0.01 and 0.001 are also considered in that paper. For comparison, the θ_δ for these values or δ and for r=0 are 0.007, 0.014, 0.030, 0.045, and 0.067. The confidence estimator is often greater than the alternative estimators and always greater than the MLE, reflecting what the proposers call converatism. It is conservative in that it certainly overstates risk relative to the MLE, and this might be an area of application in which upside errors are less problematic than downside errors. There is no proposed or obvious estimator for the confidence one should have in the precision of the confidence estimator.

Let us confront these estimators with a stress test, in the spirit of the validation exercises expected of financial institutions OCC (2006). Table 2 reports the same complement of estimates and standard errors, now for a sample of size 10 and a sample of size 1000.

Table 2: Estimates and associated standard errors, Stress Test

n	r	$\overline{\theta}$	σ_θ	$\overline{\theta}_{lce}$	$\sigma_{\theta_{lce}}$	$\widehat{\theta}$	$\sigma_{\widehat{\theta}}$	$\theta_{0.1}$
10	0	0.0102	0.0039	0.0093	0.0075	0	0	0.2057
10	1	0.0118	0.0044	0.0155	0.0097	0.1	0.0949	0.3368
10	2	0.0133	0.0044	0.0217	0.0114	0.2	0.1265	0.4496
1000	0	0.0041	0.0016	0.0013	0.0011	0	0	0.0023
1000	10	0.0102	0.0025	0.0100	0.0029	0.01	0.0031	0.0154
1000	50	0.0343	0.0045	0.0447	0.0061	0.05	0.0069	0.0600

Notes: The posterior mean and standard deviation, the same based on a less confident expert, the MLE and its s.e., the confidence estimator.

For the very small sample size the probability estimators both appear reasonable. The preferred estimator is the one that actually reflects expert judgement, given in columns 3 and 4. The MLE is very sensitive to the number of defaults and gives the estimator zero for zero-default samples. The confidence estimator appears preposterous. For very large samples all estimators work better, as expected, though the experts pull estimators toward the prior means while the confidence estimator is regularly above the likelihood estimator. The MLE for r=0 is again a potential problem, but this is a very unlikely sample for n=1000.

Two final comments on the confidence estimator are appropriate. First, consider an example in which a sample of 100 observations is drawn from a Binomial(100,0.01) distribution. A zero will be observed with probability 0.366; the MLE will be zero and $\theta_{0.1}$ will be positive. This might be considered desirable, though our expert might regard the estimate 0.023 (see Table 1) quite surprising. It is, after all, more than twice the true value in our example. With probability 0.634, the number of defaults will be positive and the confidence estimator will substantially overestimate the true value of the default probability. Second, consider a model for nondefaults - survival. In our samples with r defaults, there are $n - r$ survivals. One might consider estimating a survival probability ρ rather than the default probability θ. Of course, $\rho = 1 - \theta$, so the associated prior is obtained by a simple linear change of variables. In our example, if the prior on θ has parameters (α, β), then the prior on ρ has parameters (β, α).There is no new prior information imposed by looking at the singular joint distribution, and there is no new data information in the distribution of $n - r$ that is not in the distribution of r. Thus, we would not in practice analyze these probabilities separately. Clearly, in the probability approach, $E(\theta|e) = 1 - E(\rho|e)$ and $E(\theta|r, e) = 1 - E(\rho|r, e)$. Since $\widehat{\rho} = (n - r)/n$, we have $\widehat{\theta} = 1 - \widehat{\rho}$ in the likelihood approach as well. Consider, however, the estimators $\theta_{0.1}$ and $\rho_{0.1}$(defined by interchanging r and n-r in 6.1). For a sample of size 100 with 5 defaults, we find $\rho_{0.1}$ =0.975 and $\theta_{0.1}$ =0.091. Referring to our discussion above (Section 2), we see that as descriptions of the uncertainties regarding defaults, the confidence estimators violate property P2, additivity, and hence are incoherent.

7 Conclusion

Expert information is crucial in risk management and specifically default modeling. This information is typically based on a mix of subjective judgement, related information not specifically modeled, and long experience with related data sets. The expert information appears in the assignment of assets to segments on the basis of their risk, in the definitions of the segments, in the choice of sample period and in the chosen statistical model. This paper argues that the expert information on the likely values of the parameters of the risk model should also be formally incorporated in the analysis.

Any measure of uncertainty should satisfy a reasonable system of properties known as coherence. The probability approach to modeling defaults is coherent, widely accepted and uncontroversial. The same justifying arguments imply that uncertainty about the unknown default rates should be modeled by probabilities. In the case of default modeling, a parametric model is customary. We use the

same approach to modeling expert information about the unknown parameters - using answers to questions about the unknown default rate to fit a parametric model to the expert's beliefs. In practice, this procedure is intensive and requires more effort and expert involvement than that given in our example. With this probability distribution in hand, updating beliefs with data information entering through the likelihood function is straightforward using Bayes Rule.
Our examples illustrate the application of the probability approach. We have used the opinions of an expert on the likely default probabilities for a risk segment in the middle of a bank's portfolio. The expert was quite certain about the likely ranges of the default probability. As a check, we also constructed a hypothetical, less-confident expert and calculated the posterior statistics. The probability approach is feasible as well as logical. We considered the likelihood approach, perhaps appropriate for very large samples and risky portfolio segments. As often noted, that approach gives an estimator of zero for default probabilities in segments with no defaults in the sample. That value is considered unacceptable. A recent proposal for estimating positive default probabilities using the upper endpoint of an approximate confidence interval for $\widehat{\theta}$ with specified coverage is also examined and is shown to be incoherent.

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